Session 12 : solutions

Let's start from

- $$s_i s_j = \begin{cases} +1 & \text{if } s_i = s_j = \pm 1 \\ 0 & \text{if } s_i \text{ or } s_j = 0 \\ -1 & \text{if } s_i = -s_j = \pm 1 \end{cases}$$

$$s_i s_j = \delta_{s_i,1}\,\delta_{s_j,1} + \delta_{s_i,-1}\,\delta_{s_j,-1} + \\ - \delta_{s_i,1}\,\delta_{s_j,(-1)} - \delta_{s_i,(-1)}\,\delta_{s_j,1}$$

- $$s_i^2 s_j^2 = \begin{cases} +1 & \text{if } s_i = \pm 1 \text{ and } s_j = \pm 1 \\ 0 & \text{if } s_i \text{ or } s_j = 0 \end{cases}$$

$$s_i^2 s_j^2 = \delta_{s_i,1}\,\delta_{s_j,1} + \delta_{s_i,-1}\,\delta_{s_j,-1} + \delta_{s_i,-1}\,\delta_{s_j,1} + \\ + \delta_{s_i,1}\,\delta_{s_j,-1} = (1 - \delta_{s_i,0})(1 - \delta_{s_j,0}) = \\ = 1 - \delta_{s_i,0} - \delta_{s_j,0} + \delta_{s_i 0}\,\delta_{s_j 0}$$

- $$s_i = \delta_{s_i,+1} - \delta_{s_i,-1}$$
- $$s_i^2 = 1 - \delta_{s_i,0}$$
- $$s_i^2 s_j = (1 - \delta_{s_i,0})(\delta_{s_j,+1} - \delta_{s_j,-1}) = \\ = \delta_{s_j,+1} - \delta_{s_j,-1} - \delta_{s_i,0}\,\delta_{s_j,+1} + \delta_{s_i,0}\,\delta_{s_j,-1}$$

Then

$$H = -J\sum_{\langle i,j\rangle}\left(\delta_{s_i,+1}\delta_{s_j,+1} + \delta_{s_i,-1}\delta_{s_j,-1} - \delta_{s_i,+1}\delta_{s_j,-1} - \delta_{s_i,-1}\delta_{s_j,+1}\right) +$$

$$+K\sum_{\langle i,j\rangle}\left(1 - \delta_{s_i,0} - \delta_{s_j,0} + \delta_{s_i 0}\delta_{s_j 0}\right) +$$

$$+G\sum_{\langle i,j\rangle}\Big(-\delta_{s_j,+1} - \delta_{s_j,-1} - \delta_{s_i,0}\delta_{s_j,+1} + \delta_{s_i,0}\delta_{s_j,-1} +$$

$$-\delta_{s_i,+1} - \delta_{s_i,-1} - \delta_{s_j,0}\delta_{s_i,+1} + \delta_{s_j,0}\delta_{s_i,-1}\Big) +$$

$$-h\sum_i\left(\delta_{s_i,+1} - \delta_{s_i,-1}\right) + D\sum_i\left(1 - \delta_{s_i,0}\right) =$$

$$= \sum_{\langle i,j\rangle}\Big\{ -J\left(\delta_{s_i,+1}\delta_{s_j,+1} + \delta_{s_i,-1}\delta_{s_j,-1}\right) +$$

$$+ J\left(\delta_{s_i,+1}\delta_{s_j,-1} + \delta_{s_i,-1}\delta_{s_j,+1}\right) + K\,\delta_{s_i,0}\delta_{s_j,0} +$$

$$- G\,\delta_{s_i,0}\delta_{s_j,+1} + G\,\delta_{s_i,0}\delta_{s_j,-1} - G\,\delta_{s_i,+1}\delta_{s_j,0} +$$

$$+ G\,\delta_{s_i,-1}\delta_{s_j,0}\Big\} +$$

$$+ \sum_i\Big\{\left(-zK - D\right)\delta_{s_i,0} - \left(h + zG\right)\delta_{s_i,+1} +$$

$$+ \left(h - zG\right)\delta_{s_i,-1}\Big\} =$$

$$= \sum_{\langle i,j \rangle} \sum_{s=\pm1,0} \sum_{s'=\pm1,0} J(s,s')\, \delta_{\sigma_i,s}\, \delta_{\sigma_j,s'} +$$

$$+ \sum_{i} \sum_{s=\pm1,0} h(s)\, \delta_{\sigma_i,s}$$

with

$$J(s,s') = \begin{array}{c c} & \begin{array}{ccc} +1 & 0 & -1 \end{array} \\ \begin{array}{c} +1 \\ 0 \\ -1 \end{array} & \begin{pmatrix} -J & -G & J \\ -G & K & G \\ J & G & -J \end{pmatrix} \end{array}$$

$$h(s) = \begin{array}{c c} \begin{array}{c} +1 \\ 0 \\ -1 \end{array} & \begin{pmatrix} -h-2G \\ -2K-D \\ +h-2G \end{pmatrix} \end{array}$$

This is a generalized Potts model, very much similar to the ones used to study machine learning algorithms (e.g. the precursors of AlphaFold.